Efficient and Near-Optimal Online Portfolio Selection

Dmitrii M. Ostrovskii

Rémi Jézéquel   Pierre Gaillard

arXiv:2209.13932

Optimization and Statistical Learning

Les Houches, France

January 20, 2023

Outline

- Problem: **Online portfolio selection.**
- Classical algorithm: **Universal Portfolios**.
- New algorithm: **VB-FTRL**.

Online portfolio problem

"*The single most iconic online learning problems.*" Gergely Neu's Twitter.

- Let Δ_d be the probability simplex, and $\mathbb{R}^d_+$ the nonnegative orthant.

Online portfolio selection (Cover, 1991)

- $\mathsf{Cap}_0 = 1$
- **For** $t \in [T] := \{1, 2, ..., T\}$ **do**
  - Select $w_t \in \Delta_d$ // distribute money over d assets
  - Receive $x_t \in \mathbb{R}^d_+$ // obtain new asset returns

$$\mathsf{Cap}_t := \underbrace{x_t^\top w_t}_{\text{overall return at round } t} \mathsf{Cap}_{t-1}.$$

Online portfolio problem

"The single most iconic online learning problems." Gergely Neu's Twitter.

- Let Δ_d be the probability simplex, and $\mathbb{R}_+^d$ the nonnegative orthant.

Online portfolio selection (Cover, 1991)

- $\text{Cap}_0 = 1$
- **For** $t \in [T] := \{1, 2, ..., T\}$ **do**
  - Select $w_t \in \Delta_d$ // distribute money over d assets
  - Receive $x_t \in \mathbb{R}_+^d$ // obtain new asset returns

$$\text{Cap}_t := \underbrace{x_t^\top w_t}_{\text{overall return at round } t} \text{Cap}_{t-1}.$$

- "Money compounds **multiplicatively**." "Market plays **against** you."
- **Goal:** choose $w_1, ..., w_T$ to earn large final capital $\text{Cap}_T = \prod_{t \in [T]} x_t^\top w_t$.

Online portfolio problem (cont.)

"The single most iconic online learning problems." Gergely Neu's Twitter.

- Let Δ_d be the probability simplex, and $\mathbb{R}^d_+$ the nonnegative orthant.

Online portfolio selection (Cover, 1991)

- $\text{Cap}_0 = 1$
- **For** $t \in [T] := \{1, 2, ..., T\}$ **do**
  - Select $w_t \in \Delta_d$ // distribute money over d assets
  - Receive $x_t \in \mathbb{R}^d_+$ // obtain new asset returns

$$\text{Cap}_t := \underbrace{x_t^\top w_t}_{\text{overall return at round } t} \text{Cap}_{t-1}.$$

- **Individual losses**:

$$\ell_t(w) := -\log(x_t^\top w),$$

maximizing $\text{Cap}_T \Leftrightarrow$ minimizing the final **cumulative loss**

$$-\log(\text{Cap}_T) = \sum_{t \in [T]} \ell_t(w_t).$$

Online optimization formulation

- **Baseline:** the **offline-best static** ("constantly-rebalanced") portfolio.

Regret of $w_{1:T}$ given $x_{1:T}$:

$$\mathscr{R}_T(w_{1:T}|x_{1:T}) := \sum_{t\in[T]} \ell_t(w_t) - \min_{w\in\Delta_d} \sum_{t\in[T]} \ell_t(w).$$

- **Goal:** guarantee small regret **uniformly** over all possible markets:

$$\sup_{x_1,\dots,x_T\in\mathbb{R}^d_+} \mathscr{R}_T(w_{1:T}|x_{1:T}) = o_d(T).$$

Cover's algorithm

Follow-The-Leader (FTL)

$$w_t \in \operatorname*{Argmin}_{w \in \Delta_d} \sum_{\tau \in [t-1]} \ell_\tau(w),$$

i.e. select the best portfolio for the observed market.

- **Fails:** for $(x_1, x_2, x_3, x_4, \ldots) = (e_1, e_2, e_1, e_2, \ldots)$ in $\mathbb{R}^2$ the regret is $\Omega(T)$.

Cover's algorithm

Follow-The-Leader (FTL)

$$w_t \in \operatorname*{Argmin}_{w \in \Delta_d} \sum_{\tau \in [t-1]} \ell_\tau(w),$$

i.e. select the best portfolio for the observed market.

- **Fails:** for $(x_1, x_2, x_3, x_4, \dots) = (e_1, e_2, e_1, e_2, \dots)$ in $\mathbb{R}^2$ the regret is $\Omega(T)$.
- **Key trick:** robustify the procedure by exponential weighting:

Universal Portfolios (a.k.a. Exponentially Weighted Average Forecaster)

Play $w_t = \mathbb{E}_{\phi_t} w$ where averaging is over the **distribution** with density

$$\phi_t(w) \propto \exp\left(-\sum_{\tau \in [t-1]} \ell_\tau(w)\right), \quad w \in \Delta_d.$$

English: "to each $w \in \Delta_d$ assign the weight proportional to the amount of money it would have earned on the observed market; take the average."

Cover's algorithm (cont.)

Theorem (Cover, 1991)

For any realization $x_{1:T}$ of the market, the regret for Universal Portfolios is

$$O(\, d \log(eT) \,).$$

Regret guarantee:

- Linear in d.
- Affine-invariant; in particular, independent on the **magnitudes** of x_t's.
- Essentially unimprovable when $T \gtrsim d$ (Cesa-Bianchi and Lugosi, 2006).

Cover's algorithm (cont.)

Theorem (Cover, 1991)

For any realization $x_{1:T}$ of the market, the regret for Universal Portfolios is

$$O(\ d \log(eT)\ ).$$

Regret guarantee:

- Linear in d.
- Affine-invariant; in particular, independent on the **magnitudes** of x_t's.
- Essentially unimprovable when $T \gtrsim d$ (Cesa-Bianchi and Lugosi, 2006).

Computationally prohibitive: amounts to integration over ϕ_t.

- ϕ_t is **log-concave** $\Rightarrow$ can sample from it to approximate $\mathbb{E}_{\phi_t}[w]$.
- Kalai and Vempala (2002): runtime $O(d^4(T+d)^{14})$ per round.

Open problem

Propose a **regret-optimal** and **computationally feasible** algorithm.

Progress timeline

Open problem

Propose a **regret-optimal** and **computationally feasible** algorithm.

Algorithm	Regret	Runtime	Source
Universal Portfolios	$d\log(T)$	$\exp(d, T)$	Cover (1991)
		$d^4 T^{14}$	Kalai and Vempala (2002)
Online Grad. Descent	$G\sqrt{Td}$	d	Zinkevich (2003)
Exponentiated Grad.	$G\sqrt{T\log(d)}$	d	Helmbold et al. (1998)
Online Newton Step	$Gd\log(T)$	d^3	Hazan et al. (2007)
Soft-Bayes	$\sqrt{Td\log(d)}$	d	Orseau et al. (2017)
Ada-BARRONS	$d^2\log^p(T)$	$d^{2.5}(T)$	Luo et al. (2018)
BISONS	$d^2\log^p(T)$	$\mathrm{poly}(d)$	Zimmert et al. (2022)
AdaMix+DONS	$d^2\log^p(T)$	$d^3\log^p(T)$	Mhammedi and Rakhlin (2022)
VB-FTRL	$d\log(T)$	$d^2 T$	**Our result**

Log-barrier regularization

- Regularize the observed losses with the **log-barrier** of Δ_d, i.e. consider

$$L_t(w) := \sum_{\tau \in [t-1]} \ell_t(w) \; - \; \lambda \sum_{i \in [d]} \log(w[i]) \quad \text{with } \lambda > 0.$$

LB-FTRL – Log-Barrier Follow-The-Regularized-Leader

$$w_t = \operatorname*{argmin}_{w \in \Delta_d} L_t(w).$$

- **Self-concordant** (SC) $\Rightarrow$ Newton's method, in $O(d^2 T)$ per step.
- Van Erven et al. (2020): $O(\sqrt{dT \log T})$ regret for $\lambda \approx \sqrt{\frac{T+d}{d}} \gg 1$
- **Conjectured** $O(d \log T)$ regret for $\lambda = O(1)$, but it was **disproved** by Zimmert et al. (2022):

$$\mathscr{R}_T(w_{1:T}|x_{1:T}) \gtrsim \mathbf{2^d} \log(T) \log\log(T) \;\; \text{when} \;\; T \gtrsim \text{poly}(2^d).$$

Key ingredient: volumetric barrier

It suffices to add to $L_{t-1}(w)$ a **volumetric** regularizer:

$$V_t(w) := \frac{1}{2} \log \det[\nabla^2 L_t(w)]$$

—**volumetric barrier** for the set of "observed linear constraints:"

$$\{w \in \Delta_d : \quad x_\tau^\top w > 0, \quad \forall \tau \in [t-1]\}.$$

- First studied by Vaidya (1989) as a **self-concordant barrier** for a polytope, improved compared to the logarithmic barrier, i.e. $L_t(w)$.

VB-FTRL – Follow-The-Regularized-Leader with a Volumetric Barrier:

$$w_t = \operatorname*{argmin}_{w \in \Delta_d} \underbrace{L_t(w) + \mu V_t(w)}_{P_t(w)}.$$

- Vaidya (1989): P_{t-1} is strictly convex and **self-concordant**:

$$|\nabla^3 P_t(w)[u,u,u]| \leqslant O(1) \cdot (\nabla^2 P_t(w)[u,u])^{3/2}.$$

Main result

In terms of the "hybrid" barrier $P_t(w) := L_t(w) + \mu V_t(w)$ we have

$$w_t = \operatorname*{argmin}_{w \in \Delta_d} P_t(w).$$

- VB-FTRL is **near-optimal** in terms of regret:

Theorem 1

For any $x_{1:T}$, VB-FTRL run with $\lambda = 16$, $\mu = 7$ produces $w_{1:T}$ such that

$$\mathscr{R}_T(w_{1:T}|x_{1:T}) \leqslant 30\,(d-1)\log(T + 16d).$$

Main result

In terms of the "hybrid" barrier $P_t(w) := L_t(w) + \mu V_t(w)$ we have

$$w_t = \operatorname*{argmin}_{w \in \Delta_d} P_t(w).$$

- VB-FTRL is **near-optimal** in terms of regret:

Theorem 1

For any $x_{1:T}$, VB-FTRL run with $\lambda = 16$, $\mu = 7$ produces $w_{1:T}$ such that

$$\mathscr{R}_T(w_{1:T}|x_{1:T}) \leqslant 30\,(d-1)\log(T+16d).$$

- VB-FTRL is **computationally cheap**:

Theorem 2

A quasi-Newton method, *when run from* w_{t-1}, converges linearly to w_t.

- Key step: prove that w_{t-1} remains in an $O(1)$-Dikin ellipsoid of w_t.

One quasi-Newton step for P_{t-1} can be performed in $O(d^2T + d^3)$.

Connection of VB-FTRL with Universal Portfolios

Variational characterization of Universal Portfolios

Universal Portfolios: play $w_t = \mathbb{E}_{w \sim \phi_t}[w]$ with ϕ_t the Gibbs distribution:

$$\phi_t(w) \propto \exp\left(-\frac{1}{\mu} \sum_{\tau \in [t-1]} \ell_\tau(w) \right), \quad w \in \Delta_d.$$

Variational characterization

Let $\mathrm{Supp}(\Delta_d)$ be the set of all distributions supported on Δ_d, and define

$$\mathcal{F}_t[\phi] := \mathcal{L}_t[\phi] - \mu \mathcal{H}[\phi]$$

where $\mathcal{L}_t[\phi] = \mathop{\mathbb{E}}_{w \sim \phi} L_t(w)$, and $\mathcal{H}[\phi]$ is the **differential entropy** of ϕ, i.e.

$$\mathcal{H}[\phi] := \mathop{\mathbb{E}}_{w \sim \phi} [-\log \phi(w)].$$

Then

$$\phi_t = \mathop{\mathrm{argmin}}_{\phi \in \mathrm{Supp}(\Delta_d)} \mathcal{F}_t[\phi].$$

- λ (inside L_t) biases the leader towards the Dirac on $\frac{1}{d}\mathbb{1}_d$.
- μ—temperature—repeals us from the leader (gives a spread around).

Optimization viewpoint

$$\min_{\phi \in \mathrm{Supp}(\Delta_d)} \{\mathcal{F}_t[\phi] \ := \ \mathcal{L}_t[\phi] - \mu\mathcal{H}[\phi]\}$$

is a **convex** optimization problem—but infinite-dimensional one, thus hard.

- Even **evaluating** $\mathcal{F}_t[\phi]$ is already **challenging** (expectation over ϕ).

Minimize approximately over a smaller class of distributions.

Outline

1°. Replace $L_t(w)$ with its quadratic approximation near $\hat{w} = \mathbb{E}_\phi[w]$.

2°. By self-concordance, this approximation is valid in a Dikin ellipsoid of L_t around $\hat{w} \Longrightarrow$ focus on the densitites **supported** in this ellipsoid.

3°. The ellipsoid constraint suggests to focus on Gaussian distributions.

Approximation of the cumulative loss via self-concordance

Let $\mathbb{A}_d = \{w \in \mathbb{R}^d : \mathbb{1}^\top w = 1\}$. From now on, we omit the time index t.

Definition: r-Dikin ellipsoid of L at $\hat{w}$

$$\mathcal{E}_r(\hat{w}) := \Big\{ w \in \mathbb{A}_d : \; \|w - \hat{w}\|_{\nabla^2 L(\hat{w})} < r \Big\}.$$

- L is a SC barrier on Δ_d, so $\mathcal{E}_1(\hat{w}) \in \Delta_d$ for any $\hat{w} \in \Delta_d$.[1]

Approximation of the cumulative loss via self-concordance

Let $\mathbb{A}_d = \{w \in \mathbb{R}^d : \mathbb{1}^\top w = 1\}$. From now on, we omit the time index t.

Definition: r-Dikin ellipsoid of L at $\hat{w}$

$$\mathcal{E}_r(\hat{w}) := \Big\{ w \in \mathbb{A}_d : \; \|w - \hat{w}\|_{\nabla^2 L(\hat{w})} < r \Big\}.$$

- Controlled quadratic approximation for all $\hat{w} \in \Delta_d$ and $w \in \mathcal{E}_{1/2}(\hat{w})$:

$$\frac{1}{5}\,\|w - \hat{w}\|^2_{\nabla^2 L_t(\hat{w})} \leqslant L(w) - L(\hat{w}) - \nabla L(\hat{w})^\top (w - \hat{w}) \leqslant \frac{4}{5}\,\|w - \hat{w}\|^2_{\nabla^2 L(\hat{w})}.$$

- Hence, for any ϕ with $\mathbb{E}_\phi w = \hat{w}$ and supported on $\mathcal{E}_{1/2}(\hat{w})$,

$$\underbrace{L(\hat{w}) + \tfrac{1}{5}\left\langle \mathrm{Cov}[\phi], \nabla^2 L(\hat{w}) \right\rangle}_{\underline{\mathcal{L}}[\phi]} \leqslant \mathcal{L}[\phi] \leqslant \underbrace{L(\hat{w}) + \tfrac{4}{5}\left\langle \mathrm{Cov}[\phi], \nabla^2 L(\hat{w}) \right\rangle}_{\bar{\mathcal{L}}[\phi]}.$$

Approximation of the cumulative loss via self-concordance

Let $\mathbb{A}_d = \{w \in \mathbb{R}^d : \mathbb{1}^\top w = 1\}$. From now on, we omit the time index t.

Definition: r-Dikin ellipsoid of L at $\hat{w}$

$$\mathcal{E}_r(\hat{w}) := \left\{ w \in \mathbb{A}_d : \; \|w - \hat{w}\|_{\nabla^2 L(\hat{w})} < r \right\}.$$

- Controlled quadratic approximation for all $\hat{w} \in \Delta_d$ and $w \in \mathcal{E}_{1/2}(\hat{w})$:

$$\frac{1}{5} \|w - \hat{w}\|^2_{\nabla^2 L_t(\hat{w})} \leqslant L(w) - L(\hat{w}) - \nabla L(\hat{w})^\top (w - \hat{w}) \leqslant \frac{4}{5} \|w - \hat{w}\|^2_{\nabla^2 L(\hat{w})}.$$

- Hence, for any ϕ with $\mathbb{E}_\phi w = \hat{w}$ and supported on $\mathcal{E}_{1/2}(\hat{w})$,

$$\underbrace{L(\hat{w}) + \frac{1}{5}\left\langle \mathrm{Cov}[\phi], \nabla^2 L(\hat{w}) \right\rangle}_{\underline{\mathcal{L}}[\phi]} \leqslant \mathcal{L}[\phi] \leqslant \underbrace{L(\hat{w}) + \frac{4}{5}\left\langle \mathrm{Cov}[\phi], \nabla^2 L(\hat{w}) \right\rangle}_{\bar{\mathcal{L}}[\phi]}.$$

- This suggests to replace $\min\limits_{\phi \in \mathrm{Supp}(\Delta_d)} \mathcal{L}[\phi] - \mu \mathcal{H}[\phi]$ with the problem

$$\min_{\substack{\hat{w} \in \Delta_d, \; \mathbb{E}_\phi[w] = \hat{w}, \\ \phi \in \mathrm{Supp}(\mathcal{E}_{1/4}(\hat{w}))}} \bar{\mathcal{L}}[\phi] - \mu \mathcal{H}[\phi].$$

Volumetric barrier appears!

$$\min_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi[w] = \hat{w}, \\ \phi \in \mathrm{Supp}(\mathcal{E}_{1/4}(\hat{w}))}} \bar{\mathcal{L}}[\phi] - \mu \mathcal{H}[\phi].$$

We can recover $\log\det[L_t(\hat{w})]$ as the negative entropy of the Gaussian distribution with optimal covariance satisfying the "soft" constraint

$$\mathrm{Cov}[\phi] \preccurlyeq \nabla^2 L_t[\hat{w}]^{-1}.$$

Dikin approximation accuracy

Let $\phi^\star$ be the Gibbs distribution, and $\bar{\phi}$ be its "upper" approximation:

$$\phi^\star := \underset{\phi \in \mathsf{Supp}(\Delta_d)}{\operatorname{argmin}} \underbrace{\mathcal{L}[\phi] - \mu\mathcal{H}[\phi]}_{\mathcal{F}[\phi]} \quad \text{and} \quad \bar{\phi} \in \underset{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}}{\operatorname{Argmin}} \underbrace{\bar{\mathcal{L}}[\phi] - \mu\mathcal{H}[\phi]}_{\bar{\mathcal{F}}[\phi]},$$

Dikin approximation accuracy

Let $\phi^\star$ be the Gibbs distribution, and $\bar{\phi}$ be its "upper" approximation:

$$\phi^\star := \underset{\phi \in \mathsf{Supp}(\Delta_d)}{\operatorname{argmin}} \underbrace{\mathcal{L}[\phi] - \mu\mathcal{H}[\phi]}_{\mathcal{F}[\phi]} \quad \text{and} \quad \bar{\phi} \in \underset{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}}{\operatorname{Argmin}} \underbrace{\bar{\mathcal{L}}[\phi] - \mu\mathcal{H}[\phi]}_{\bar{\mathcal{F}}[\phi]},$$

- As it turns out, $\bar{\phi}$ is not much worse than ϕ^* in terms of $\mathcal{F}[\cdot]$, namely:

Proposition 1

For any $\lambda \geqslant 1$ and $\mu \geqslant 0$,

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \min_{\phi \in \mathsf{Supp}(\Delta_d)} \mathcal{F}[\phi] + 1.5\mu(d-1)\log(T + \lambda d) + C\mu + c.$$

- The first inequality is trivial. I shall outline the proof of the second one.

Dikin approximation accuracy: proof sketch (I)

$$\bar{\phi} \in \mathop{\mathrm{Argmin}}_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}} \underbrace{\bar{\mathcal{L}}[\phi] - \mu \mathcal{H}[\phi]}_{\bar{\mathcal{F}}[\phi]}. \qquad (\bar{\mathsf{P}})$$

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \min_{\phi \in \mathsf{Supp}(\Delta_d)} \mathcal{F}[\phi] + 1.5\mu(d-1)\log(T + \lambda d) + C\mu + c.$$

Dikin approximation accuracy: proof sketch (I)

$$\bar{\phi} \in \underset{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w},\\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}}{\mathrm{Argmin}} \underbrace{\bar{\mathcal{L}}[\phi] - \mu\mathcal{H}[\phi]}_{\bar{\mathcal{F}}[\phi]}. \qquad (\bar{\mathsf{P}})$$

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \min_{\phi \in \mathsf{Supp}(\Delta_d)} \mathcal{F}[\phi] + 1.5\mu(d-1)\log(T + \lambda d) + C\mu + c.$$

- Recall that for $\phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))$ we also have $\mathcal{F}[\phi] \geqslant \underline{\mathcal{F}}[\phi]$, and let

$$\underline{\phi} \in \underset{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w},\\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))}}{\mathrm{Argmin}} \underbrace{\underline{\mathcal{L}}[\phi] - \mu\mathcal{H}[\phi]}_{\underline{\mathcal{F}}[\phi]}.$$

Dikin approximation accuracy: proof sketch (I)

$$\bar{\phi} \in \underset{\substack{\hat{w} \in \Delta_d, \; \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}}{\mathrm{Argmin}} \underbrace{\bar{\mathcal{L}}[\phi] - \mu \mathcal{H}[\phi]}_{\bar{\mathcal{F}}[\phi]}. \qquad (\bar{\mathsf{P}})$$

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \min_{\phi \in \mathsf{Supp}(\Delta_d)} \mathcal{F}[\phi] + 1.5\mu(d-1)\log(T + \lambda d) + C\mu + c.$$

- Recall that for $\phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))$ we also have $\mathcal{F}[\phi] \geqslant \underline{\mathcal{F}}[\phi]$, and let

$$\underline{\phi} \in \underset{\substack{\hat{w} \in \Delta_d, \; \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))}}{\mathrm{Argmin}} \underbrace{\underline{\mathcal{L}}[\phi] - \mu \mathcal{H}[\phi]}_{\underline{\mathcal{F}}[\phi]}.$$

- Moreover, let $\underline{\phi}^+(\cdot) = 2\underline{\phi}(\underline{w} + 2(\cdot - \underline{w}))$ be the "squeezing-by-factor-2" of $\underline{\phi}$ towards $\underline{w} = \mathbb{E}_{\underline{\phi}}[w]$, i.e. the distribution of $\underline{w} + \frac{1}{2}(w - \underline{w})$ for $w \sim \underline{\phi}$; then

$$\mathbb{E}_{\underline{\phi}^+}[w] = \mathbb{E}_{\underline{\phi}}[w] = \underline{w} \quad \text{and} \quad \underline{\phi}^+ \in \mathsf{Supp}(\mathcal{E}_{1/4}(\underline{w})).$$

Dikin approximation accuracy: proof sketch (I)

$$\bar{\phi} \in \underset{\substack{\hat{w} \in \Delta_d,\; \mathbb{E}_\phi w = \hat{w},\\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}}{\mathrm{Argmin}} \underbrace{\bar{\mathcal{L}}[\phi] - \mu\mathcal{H}[\phi]}_{\bar{\mathcal{F}}[\phi]}. \qquad (\bar{\mathsf{P}})$$

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \min_{\phi \in \mathsf{Supp}(\Delta_d)} \mathcal{F}[\phi] + 1.5\mu(d-1)\log(T+\lambda d) + C\mu + c.$$

- Recall that for $\phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))$ we also have $\mathcal{F}[\phi] \geqslant \underline{\mathcal{F}}[\phi]$, and let

$$\underline{\phi} \in \underset{\substack{\hat{w} \in \Delta_d,\; \mathbb{E}_\phi w = \hat{w},\\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))}}{\mathrm{Argmin}} \underbrace{\underline{\mathcal{L}}[\phi] - \mu\mathcal{H}[\phi]}_{\underline{\mathcal{F}}[\phi]}.$$

- Moreover, let $\underline{\phi}^+(\cdot) = 2\underline{\phi}(\underline{w} + 2(\cdot - \underline{w}))$ be the "squeezing-by-factor-2" of $\underline{\phi}$ towards $\underline{w} = \mathbb{E}_{\underline{\phi}}[w]$, i.e. the distribution of $\underline{w} + \frac{1}{2}(w - \underline{w})$ for $w \sim \underline{\phi}$; then

$$\mathbb{E}_{\underline{\phi}^+}[w] = \mathbb{E}_{\underline{\phi}}[w] = \underline{w} \quad \text{and} \quad \underline{\phi}^+ \in \mathsf{Supp}(\mathcal{E}_{1/4}(\underline{w})).$$

- In other words, $(\underline{w}, \underline{\phi}^+)$ is **feasible** in $(\bar{\mathsf{P}})$, and therefore $\bar{\mathcal{F}}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\underline{\phi}^+]$
- But we also have $\mathrm{Cov}[\underline{\phi}^+] = \frac{1}{4}\mathrm{Cov}[\underline{\phi}]$ and $\mathcal{H}[\underline{\phi}^+] = \mathcal{H}[\underline{\phi}] - (d-1)\log(2)$, so

$$\bar{\mathcal{F}}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\underline{\phi}^+] \leqslant \underline{\mathcal{F}}[\underline{\phi}] + \mu(d-1)\log(2).$$

Dikin approximation accuracy: proof sketch (II)

So far, for

$$\bar{\phi} \in \operatorname*{Argmin}_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/4}(\hat{w}))}} \bar{\mathcal{F}}[\phi] \quad \text{and} \quad \underline{\phi} \in \operatorname*{Argmin}_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathsf{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \underline{\mathcal{F}}[\phi]$$

we have the following:

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \underline{\mathcal{F}}[\underline{\phi}] + \mu(d-1)\log(2).$$

Dikin approximation accuracy: proof sketch (II)

So far, for

$$\bar{\phi} \in \operatorname*{Argmin}_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathrm{Supp}(\mathcal{E}_{1/4}(\hat{w}))}} \bar{\mathcal{F}}[\phi] \quad \text{and} \quad \underline{\phi} \in \operatorname*{Argmin}_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathrm{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \underline{\mathcal{F}}[\phi]$$

we have the following:

$$\mathcal{F}[\bar{\phi}] \leqslant \bar{\mathcal{F}}[\bar{\phi}] \leqslant \underline{\mathcal{F}}[\underline{\phi}] + \mu(d-1)\log(2).$$

- But since $\underline{\mathcal{F}}[\phi] \leqslant \mathcal{F}[\phi]$ for all $\phi \in \mathrm{Supp}(\mathcal{E}_{1/2}(\hat{w}))$, we conclude that

$$\mathcal{F}[\bar{\phi}] \leqslant \min_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathrm{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \mathcal{F}[\phi] \;+\; \mu(d-1)\log(2).$$

- We've gotten rid of the **objective approximation.** It remains to prove that

$$\min_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathrm{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \mathcal{F}[\phi] \leqslant \min_{\phi \in \mathrm{Supp}(\Delta_d)} \mathcal{F}[\phi] + O(\,\mu(d-1)\log(T+\lambda d) + \mu + 1\,).$$

Open questions and perspectives

- Reduce complexity $O(d^2 T + d^3)$ to $O(d^3)$?
- **Quantum state estimation:** instead of $w \in \Delta_d$ consider

$$W \in \mathbb{S}_+^d \quad \mathrm{tr}[W] = 1.$$

- Comparing with the best **distribution** $\phi^* \in \Delta_d$?
- Using "truncated Laplace approximation" as a tool in other online learning problems with **expensive**/intractable optimal strategies:
  - Coding convex bodies (cf. Mourtada)?
  - Online linear optimization with bandit feedback?
  - [...]
- Accuracy of the Laplace approximation (cf. Katsevich)?

References I

Nicolo Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games*. Cambridge university press, 2006.

Thomas M. Cover. Universal portfolios. *Mathematical Finance*, 1(1):1–29, 1991.

Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2):169–192, 2007.

David P. Helmbold, Robert E. Schapire, Yoram Singer, and Manfred K. Warmuth. On-line portfolio selection using multiplicative updates. *Mathematical Finance*, 8 (4):325–347, 1998.

Adam T. Kalai and Santosh Vempala. Efficient algorithms for universal portfolios. *Journal of Machine Learning Research*, pages 423–440, 2002.

Haipeng Luo, Chen-Yu Wei, and Kai Zheng. Efficient online portfolio with logarithmic regret. In *Proceedings of the 33rd international Conference on Neural Information Processing Systems*, volume 31, pages 8245–8255, 2018.

Zakaria Mhammedi and Alexander Rakhlin. Damped online Newton step for portfolio selection. *arXiv preprint arXiv:2202.07574*, 2022.

Yurii Nesterov. *Introductory lectures on convex optimization: A basic course*, volume 87. Springer Science & Business Media, 2003.

References II

Laurent Orseau, Tor Lattimore, and Shane Legg. Soft-bayes: Prod for mixtures of experts with log-loss. In *International Conference on Algorithmic Learning Theory*, pages 372–399. PMLR, 2017.

Pravin M. Vaidya. A new algorithm for minimizing convex functions over convex sets. In *30th Annual Symposium on Foundations of Computer Science*, pages 338–343. IEEE Computer Society, 1989.

Tim Van Erven, Dirk Van der Hoeven, Wojciech Kotłowski, and Wouter M. Koolen. Open problem: Fast and optimal online portfolio selection. In *Proceedings of the 33rd Conference On Learning Theory*, pages 3864–3869. PMLR, 2020.

Julian Zimmert, Naman Agarwal, and Satyen Kale. Pushing the efficiency-regret Pareto frontier for online learning of portfolios and quantum states. *arXiv preprint arXiv:2202.02765*, 2022.

Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the 20th International Conference on Machine Learning*, pages 928–936, 2003.

Dikin approximation accuracy: proof sketch (III)

It remains to prove that

$$\min_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w}, \\ \phi \in \mathrm{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \mathcal{F}[\phi] \leqslant \min_{\phi \in \mathrm{Supp}(\Delta_d)} \mathcal{F}[\phi] + O(\mu(d-1)\log(T+\lambda d) + \mu + 1).$$

- Note: the leader $w^\star = \underset{w \in \Delta_d}{\operatorname{argmin}}\, L(w)$ is the **mode** of $\phi^\star = \underset{\phi \in \mathrm{Supp}(\Delta_d)}{\operatorname{argmin}} \mathcal{F}[\phi]$.
- Now, let ϕ^{trc} be the truncation of $\phi^\star$ to $\mathcal{E}_{1/8}(w^\star)$, that is

$$\phi^{\mathrm{trc}}(w) = \frac{\exp\left(-\frac{1}{\mu}L(w)\right)}{\int_{\mathcal{E}_{1/8}(w^\star)} \exp\left(-\frac{1}{\mu}L(w')\right) dw'}, \quad w \in \mathcal{E}_{1/8}(w^\star).$$

Dikin approximation accuracy: proof sketch (III)

It remains to prove that

$$\min_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w},\\ \phi \in \mathrm{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \mathcal{F}[\phi] \leqslant \min_{\phi \in \mathrm{Supp}(\Delta_d)} \mathcal{F}[\phi] + O(\,\mu(d-1)\log(T+\lambda d) + \mu + 1\,).$$

- Note: the leader $w^\star = \operatorname*{argmin}_{w \in \Delta_d} L(w)$ is the **mode** of $\phi^\star = \operatorname*{argmin}_{\phi \in \mathrm{Supp}(\Delta_d)} \mathcal{F}[\phi]$.
- Now, let ϕ^{trc} be the truncation of $\phi^\star$ to $\mathcal{E}_{1/8}(w^\star)$, that is

$$\phi^{\mathrm{trc}}(w) = \frac{\exp\left(-\frac{1}{\mu}L(w)\right)}{\int_{\mathcal{E}_{1/8}(w^\star)} \exp\left(-\frac{1}{\mu}L(w')\right) dw'}, \quad w \in \mathcal{E}_{1/8}(w^\star).$$

- By self-concordance and triangle ineq., $\phi^{\mathrm{trc}} \in \mathcal{E}_{1/2}(w^{\mathrm{trc}})$ with $w^{\mathrm{trc}} = \mathbb{E}_{\phi^{\mathrm{trc}}}[w]$.
- In other words, ϕ^{trc} is feasible in the restricted minimization problem, and

$$\min_{\substack{\hat{w} \in \Delta_d,\ \mathbb{E}_\phi w = \hat{w},\\ \phi \in \mathrm{Supp}(\mathcal{E}_{1/2}(\hat{w}))}} \mathcal{F}[\phi] \leqslant \mathcal{F}[\phi^{\mathrm{trc}}].$$

- It remains to show that $\mathcal{F}[\phi^{\mathrm{trc}}]$ is not much larger than $\mathcal{F}[\phi^\star]$.

Dikin approximation accuracy: proof sketch (IV)

Truncation lemma

As long as $\lambda \geqslant 1$, for

$$\phi^\star(w) = \frac{\exp\left(-\frac{1}{\mu}L(w)\right)}{\int_{\Delta_d} \exp\left(-\frac{1}{\mu}L(w')\right) dw'} \quad \text{and} \quad \phi^{\text{trc}}(w) = \frac{\exp\left(-\frac{1}{\mu}L(w)\right)}{\int_{\mathcal{E}_{1/8}(w^\star)} \exp\left(-\frac{1}{\mu}L(w')\right) dw'}$$

we have that

$$\mathcal{F}[\phi^{\text{trc}}] \leqslant \mathcal{F}[\phi^\star] + 1.5\mu(d-1)\log(T+\lambda d) + 3.2\mu(d+1) + 0.1.$$

Proof outline:

- By Gibbs' duality, $\mathcal{F}[\phi^\star]$ and $\mathcal{F}[\phi^{\text{trc}}]$ are the negative log-partition functions:

$$\mathcal{F}[\phi^\star] = -\mu \log\left(\int_{\Delta_d} \exp\left(-\frac{1}{\mu}L(w')\right) dw'\right),$$
$$\mathcal{F}[\phi^{\text{trc}}] = -\mu \log\left(\int_{\mathcal{E}_{1/8}(w^\star)} \exp\left(-\frac{1}{\mu}L(w')\right) dw'\right).$$

- Compare the volumes of Δ_d and $\mathcal{E}_{1/8}(w^\star)$ using the barrier property of $L(\cdot)$.