Maximization of recurrent sequences, Schur positivity [and some conjectures on polynomial interpolation]$^{\circledast}$

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based on the joint work with Pavel Shcherbakov

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$^{\circledast}$ If time permits...

Outline

Intro:

0. Brief recap of (linear, homogeneous) ODEs and classic stability.
1. Discrete-time: ordinary difference eqs, aka linear recurrences with constant coeffs. Amplitude maximization problem.
2. Formulate the main result and discuss it.

"Meat"

- Symmetric functions & Schur positivity
- Main thm proof & extensions

"Dessert"

? Further conjectures (time permitting).

0. Brief recap of ODEs and classic stability.

ODE recap

- Linear ODE of order n (with constant coefficients) on $\mathbb{R}_+$ reads:

$$x(t) + h_1 x'(t) + \ldots + h_n x^{(n)}(t) \equiv 0 \quad \Leftrightarrow \quad \boxed{h\left(\frac{d}{dt}\right) x \equiv 0} \qquad (*)$$

on $\mathbb{R}_+$

where $h(z) = 1 + h_1 z + \ldots + h_n z^n$ is the characteristic polynomial of $(*)$.

- Solution: any function $x : \mathbb{R}_+ \to \mathbb{R}$ satisfying $(*)$.

Solution set: the set of all solutions.

- The solution set of a given ODE is a linear subspace of $C^n(\mathbb{R}_+)$ of dimension $n = \deg(h)$. Namely, if h has distinct roots, then the solution set is $\operatorname{Span}\{e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\}$, where $\lambda_1, \ldots, \lambda_n$ are the reciprocal roots:

$$\boxed{h(z) = \prod_{k \in [n]} (1 - \lambda_k z)}$$

Example: $h(z) = 1 - \lambda z \Rightarrow x - \lambda x' = 0 \Leftrightarrow x(t) = Ce^{\lambda t}$

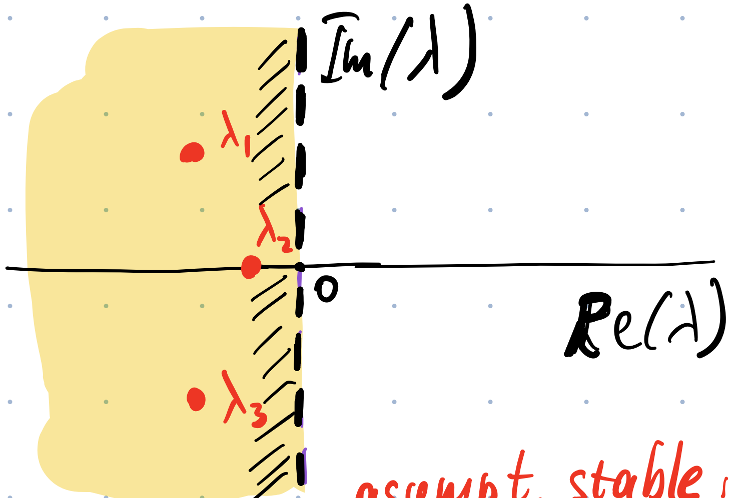
Asymptotic stability, continuous time

Consider $\boxed{h\left(\frac{d}{dt}\right)x = 0}$ with $\boxed{h(z) = \prod_{k\in[n]} (1-\lambda_k z)}$. (⊛)

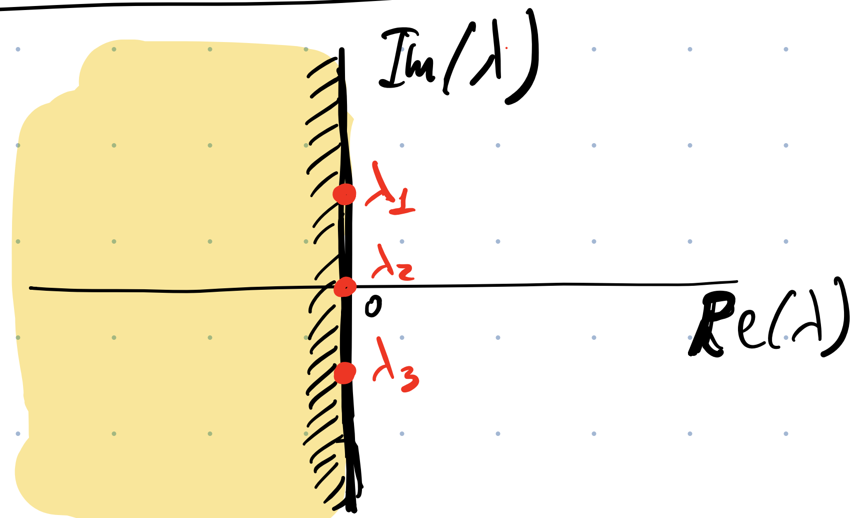
and (bounded) initial conditions: $|\alpha_k| = |x^{(k)}(0)| < \varepsilon_k$ for $k \in [n]$.

Def.: ODE (⊛) is BIBO-stable (resp., asymptotically stable) if for any initial data, one has $\limsup_{t\to+\infty} |x(t)| < +\infty$ (resp. $=0$).

⊙ Recall the classic criterion of stability:



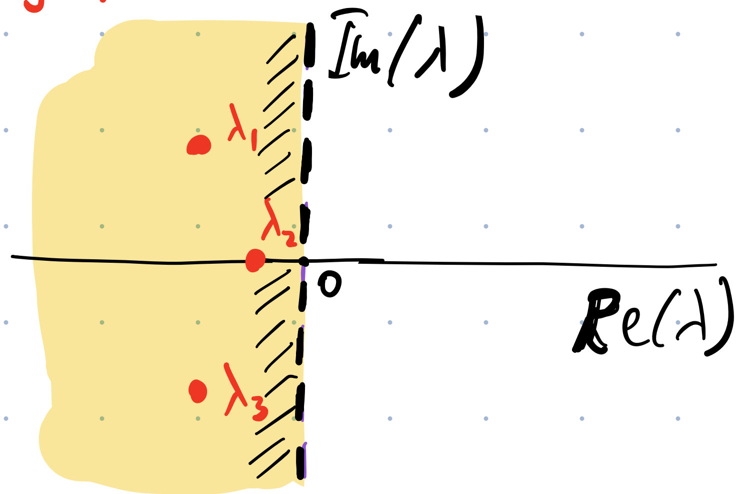
asympt. stable: $Re(\lambda_k) < 0$



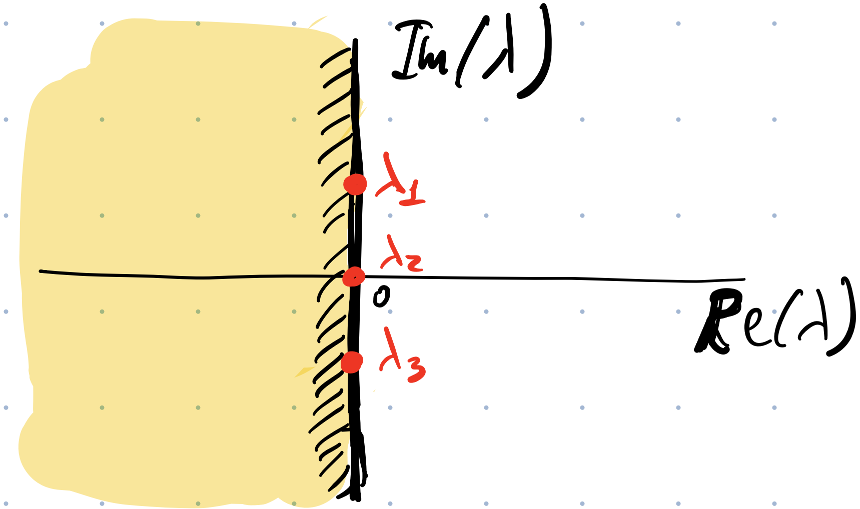
BIBO stable: $Re(\lambda_k) \le 0$, distinct at boundary.

Asymptotic stability, continuous time

asympt. stable:



BIBO stable



- Distinct roots: $$x(t) = \sum_{k\in[n]} c_k e^{\lambda_k t}$$

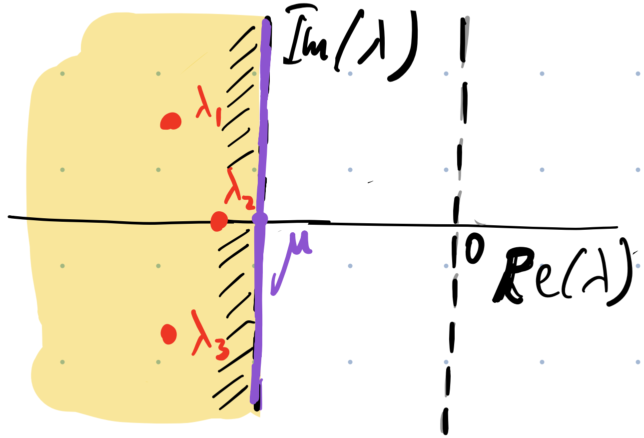
where c_k's depend on d_k's <u>linearly</u> via λ_k's. (through inverse Wronskian matrix)

Hence, $|c_k| < \infty$ and $|x(t)| \le \left(\sum_{k\in[n]} |c_k|\right) \max_{k\in[n]} e^{Re(\lambda_k)t}$.

- Equal roots result in polynomial factors, suppressed by $e^{\lambda t}$ with $Re(\lambda)<0$. ∎

Resonance is worst-case (asymptotically)

- Consider the μ-shifted half-plane:



$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq \mu < 0\}$$

- Asymptotically, complete resonance gives the worst case:

If we allow any $|d_k| \leq \varepsilon_k$, the worst case is attained with equal roots: $\lambda_k \equiv \mu$

- Indeed, this gives the largest possible degree of polynomial factor: $n-1$.

But what arbitrary t?

TLDR: as it turns out, in discrete time, complete resonance is worst-case for any $t \in \mathbb{Z}_+$ – and with the same initials. In other words, all amplitudes are maximized at once.

- We conjecture this remains true in continuous time...

1. Discrete-time: ordinary difference eqs, aka linear recurrences

Discrete time: difference equations (a.k.a. linear recurrences)

- From now on, discrete time: $t \in \mathbb{N} = \{0, 1, 2, \dots\}$
- Space of complex sequences $\mathbb{C}^{\mathbb{N}}$ and unit advance operator $A : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$:

$$[Ax]_t = x_{t+1}$$

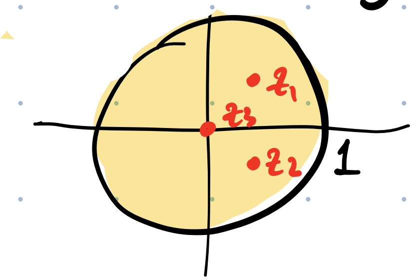
$A - I$ is the discrete derivative operator.

- Linear recurrence of order n

$$f(A)\, x \equiv 0$$

where $f(z) = z^n + f_1 z^{n-1} + \dots + f_n = (z - z_1)\dots(z - z_n)$ is monic of degree n.

- Classic stability theory has a full analogue (and is as easy to prove)

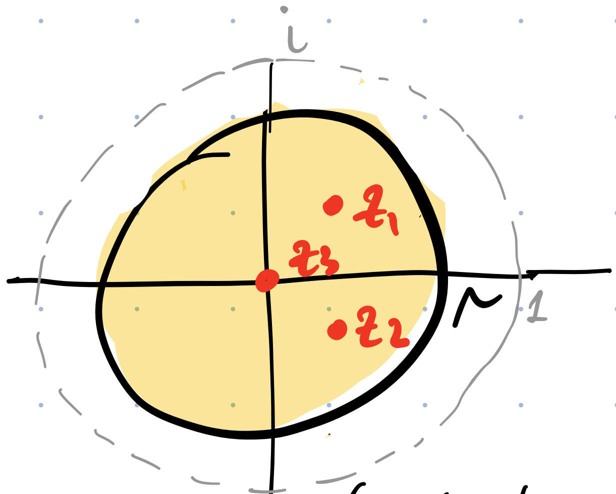


Asymp. stable: $|z_k| < 1$

Schur-stable: $|z_k| \leq 1$ (and distinct at boundary)

- Indeed, with distinct roots the general solution $x_t = \sum_{k \in [n]} c_k z_k^t$.

Asymptotic behavior under r-stability



⊙ r-stable char. polynomial ($|z_n| \le r < 1$) results in the asymptotic behavior

$$\limsup_{t\to+\infty} \log|x_t| - t\log r - (n-1)\log t < +\infty,$$

where the last term is saturated when $|z_1| = \dots = |z_n| = r$

⊙ So, asymptotically, complete resonance is again worst-case.

Time to formulate our problem ...

Amplitude maximization problem

- let: $Z_n \subset \mathbb{C}^n$ be a candidate domain for the tuple of roots: $z_{1:n} \in Z_n$.
  $P_n(Z_n)$ be the set of monic polynomials with roots in Z_n.
  $X(f)$ be the solution set of $f(A)x = 0$, for a given $f \in P_n$.
  $U_n \subset \mathbb{C}^n$ be a candidate domain for the initialization vector

- Functionals: $M_t(f \,|\, U_n) := \sup \{ |x_t| : x \in X(f),\ x_{0:n-1} \in U_n \}$.

$$M_t^*(Z_n \,|\, U_n) := \sup_{f \in P_n(Z_n)} M_t(f \,|\, U_n)$$

- We are interested in $M_t^*(Z_n \,|\, U_n)$ for a natural class of domains:
  Polydisc: $\mathbb{D}_n(r_1, .., r_n) := \mathbb{D}(r_1) \times \dots \times \mathbb{D}(r_n)$.

where $\mathbb{D}(r) = \{ z \in \mathbb{C} : |z| \le r \}$. This includes the usual disc $\mathbb{D}_n(r, .., r) = \mathbb{D}(r)^n$.

2. Formulate the main result and discuss it.

Main result

$$M_t(f \mid U_n) := \sup\{|x_t| : x \in X(f),\ x_{0:n-1} \in U_n\}$$

$$M_t^{\circledast}(Z_n \mid U_n) := \sup_{f \in P_n(Z_n)} M_t(f \mid U_n).$$

Theorem

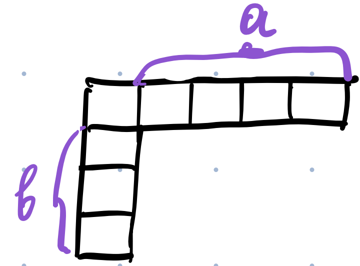
① For $n, t \in \mathbb{N}$ with $t \geq n$, arbitrary $r_{1:n} \in \mathbb{R}_+^n$ and $w_{1:n} \in \mathbb{R}_+^n$

$$\boxed{M_t^{\circledast}(\mathbb{D}_n(r_{1:n}) \mid \mathbb{D}_n(w_{1:n})) = M_t(f \mid \mathbb{D}_n(w_{1:n}))}$$

where $f(z) = (z - e^{i\theta} r_1) \cdot \ldots \cdot (z - e^{i\theta} r_n)$ is obviously in $P_n(\mathbb{D}_n(r_{1:n}))$.

② Moreover, this value is $\sum_{k \in [n]} w_k \, s_{(t-n|n-k)}(r_{1:n})$,

where $s_{(a|b)}$:= Schur polynomial of the hook shape:



Let's discuss the easy step first...

Reduction to interpolation

Maximization in $x_{0:n-1}$ reduces to the interpolation of monomials.

Lemma 1 Solution $x \in X(f)$ with initial values $x_{0:n-1}$ reads:

$$x_t = \sum_{k=0}^{n-1} \psi_k^{(t)} x_k = \langle \psi_{0:n-1}^{(t)}[z_{1:n}], x_{0:n-1} \rangle$$

where $\psi^{(t)}_{[z_{1:n}]}(\cdot)$ is the degree $n-1$ polynomial interpolating z^t on $z_{1:n}$.

⊙ As the result, for $f(z) = (z-z_1)\cdot\ldots\cdot(z-z_n)$ we are to maximize

$$M_t(f \mid \mathbb{D}_n(w_{1:n})) = \sum_{k=0}^{n-1} |\psi_k^{(t)}[z_{1:n}]|,$$

and the maximizing initials align the phases of $\psi_k^{(t)}$'s.

It "only" remains to maximize RHS in $z_{1:n} \in \mathbb{D}_n(r_{1:n})$.

Easy case: $t=n$ (instant. amplitude) ~Gautschi (1966)

- By the lemma, at $t=n$ we have $|x_t| \le \sum'_{k\in[n]} w_k \left|\psi^{(n)}_{k-1}[z_{1:n}]\right|$.
- But $\psi^{(n)}[z_{1:n}]$ is a very simple polynomial: clearly,

$$\psi^{(n)}(z) = z^n - f(z) = z^n - \prod_{k\in[n]} (z - z_k).$$

- Here $\psi^{(n)}_k = (-1)^{n-k}\, e_{n-k}(z_{1:n})$

where $\boxed{e_k(z_{1:n}) = \sum'_{1\le j_1<\ldots<j_k\le n} z_{j_1} z_{j_2} \cdots z_{j_k}}$

is the k-th elementary symmetric polynomial in $z_{1:n}$.

- The bound $|e_k(z_1, \ldots, z_n)| \le e_k(|z_1|, \ldots, |z_n|)$ is tight. ▨

This precludes the main insight.

- Symmetric functions & Schur positivity
- Main thm proof & extensions

Proof : analytical part

Recall that $\psi^{(t)}(z)$, of degree $n-1$, interpolates z^t on a simple grid $z_{1:n}$.

Lemma 2 (folklore).

Let $\hat{g}(\cdot)$, of degree $n-1$, interpolate a holomorphic $g(\cdot)$ on $z_{1:n} \in \mathbb{D}_n(r)$.

Then $\forall z \in \mathbb{D}(r)$, in terms of $f(z) = (z-z_1)\cdots(z-z_n)$, we have:

$$\boxed{g(z) - \hat{g}(z) = \frac{f(z)}{2\pi i} \oint_{|\xi| = r} \frac{g(\xi)}{f(\xi)(\xi - z)}\, d\xi}$$

Corollary Let $\max\{|z|, |z_1|, \dots, |z_n|\} \le r$. Then $\varepsilon_t(z) := z^t - \psi^{(t)}(z)$ satisfies:

$$\varepsilon_t(z) = \frac{f(z)}{2\pi i} \oint_{|\xi| = r} \frac{\xi^t}{f(\xi)(\xi - z)}\, d\xi$$

Symmetric polynomials e_k, h_k

- Elementary: $e_k(z_1, z_2, \dots z_n) = \sum_{1 \le j_1 < \dots < j_k \le n} z_{j_1} z_{j_2} \cdots z_{j_k}$,
- Complete: $h_k(z_1, z_2, \dots z_n) = \sum_{1 \le j_1 \le \dots \le j_k \le n} z_{j_1} z_{j_2} \cdots z_{j_k}$.

I.e., h_k includes all monomials; e_k only multilinear ones.

Proposition 1

If $\max\{|z_0|, |z_1|, \dots, |z_n|\} \le r$, then $\varepsilon_t(z_0) = z_0^t - \psi^{(t)}(z_0)$ satisfies

$$\boxed{\varepsilon_t(z_0)} = f(z_0)\, h_{t-n}(z_{0:n}) = \boxed{f(z_0) \sum_{d=0}^{t-n} z_0^{t-n-d}\, h_d(z_{1:n})}$$

Remark: If $z_0 = 0$, then we recover $\varepsilon_t(0) = \psi^{(t)}(0)$ and solve the amp. mex. problem for $w_{0:n-1} = (1, 0, \dots, 0)$, the unit pulse response.

Proof.

$$\frac{\varepsilon_t(z_0)}{f(z_0)} = \frac{1}{2\pi i}\oint_{|\xi|=r} \frac{\xi^t}{f(\xi)(\xi - z_0)}\,d\xi$$

$$= \frac{1}{2\pi i}\oint_{|\xi|=r} \frac{\xi^{t-n-1}}{\prod_{k=0}^{n}(1-\xi^{-1}z_k)}\,d\xi \qquad (t \geq n)$$

$$= \frac{1}{2\pi i}\oint_{|\xi|=r} \xi^{t-n-1}\prod_{k=0}^{n}\left(\sum_{s_k=0}^{+\infty} \xi^{-s_k} z_k^{s_k}\right)d\xi \qquad \text{(Geom. progr.)}$$

$$= \frac{1}{2\pi i}\oint_{|\xi|=r} \xi^{t-n-1}\sum_{d=0}^{+\infty} \xi^{-d}\, h_d(z_{0:n})\,d\xi \qquad \text{(Collect terms)}$$

$$= \sum_{d=0}^{+\infty} h_d(z_{0:n})\underbrace{\frac{1}{2\pi i}\oint_{|\xi|=r} \xi^{t-n-d-1}\,d\xi}_{\text{(Residue thm)} = \begin{cases} 1 & \text{if } t-n-d=0 \\ 0 & \text{otherwise: } \xi^{t-n-d-1} \text{ is either holomorphic or has a pole of order} > 1 \text{ at } 0. \end{cases}}$$

$\square$

Taking derivatives...

Proposition 2. Under the premise of Proposition 1, for $k \in \mathbb{N}$:

$$\frac{d^k}{dz^k} \varepsilon_t(z) \Big|_{z=0} = k! \sum_{j=0}^{\min(k,\, t-u)} (-1)^{u-k+j} e_{u-k+j}(z_{1:u})\, h_{t-u-j}(z_{1:u})$$

Proof.

$$\varepsilon_t^{(k)}(z) = \sum_{d=0}^{t-u} h_d(z_{1:u}) \left(z^{t-u-d} f(z) \right)^{(k)}$$

(Leibniz rule)

$$= \sum_{d=0}^{t-u} h_d(z_{1:u}) \sum_{j=0}^{k \wedge t-u-d} \binom{k}{j} \frac{(t-u-d)!}{(t-u-d-j)!} z^{t-u-d-j} f^{(k-j)}(z)$$

$$= k! \sum_{d=0}^{t-u} h_d(z_{1:u}) \sum_{j=0}^{k \wedge t-u-d} \binom{t-u-d}{j} z^{t-u-d-j} \frac{f^{(k-j)}(z)}{(k-j)!}$$

$$\Rightarrow \varepsilon_t^{(k)}(0) = k! \sum_{d=0}^{t-u} h_d(z_{1:u})\, \mathbb{1}\{t-u-d \le k\}\, e_{\cancel{u}-k+t-\cancel{u}-d}\, (-1)^{\cancel{u}-k+t-\cancel{u}-d}$$

▨

Symmetric functions

Best source: Stanley EC-2 book, Chap. 7.

Def. symmetric function φ in indeterminates $x_1, x_2, \dots$ is a formal power series $\varphi(x_1, x_2, \dots)$ whose terms are monomials in $x_1, x_2, \dots$ such that $\varphi(x_1, x_2, \dots) = \varphi(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$ for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$. The space of such functions is Λ.

Def.: Λ^k denotes the vector space of symm. functions of degree k

Examples:

power sum s.f.

$$e_d(z_1, z_2, \dots) = \sum_{1 \le j_1 < \dots < j_d} z_{j_1} z_{j_2} \cdots z_{j_d} \qquad p_d(z_1, z_2, \dots) = z_1^d + z_2^d + \dots$$

$$h_d(z_1, z_2, \dots) = \sum_{1 \le j_1 \le \dots \le j_d} z_{j_1} z_{j_2} \cdots z_{j_d} \qquad m_{(3,1)}(z_1, z_2, \dots) = z_1^3 z_2 + z_1^3 z_3 + \dots + z_2^3 z_1 + z_2^3 z_3 + z_2^3 z_3 + \dots$$

Specialization: for any homogeneous symmetric function g of degree k

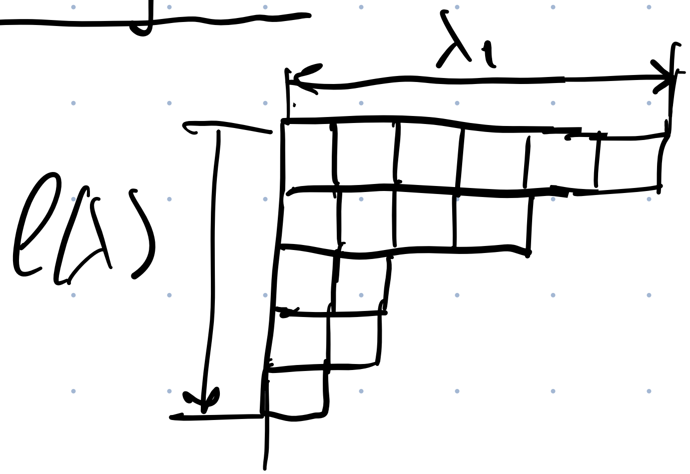
$$g(z_{1:n}) = g(z_1, \dots, z_n, 0, \dots)$$

This allows for simple calculations in the lifted space.

Monomial basis

⊙ Partition: nonnegative, nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$.
(trailing zeroes omitted)

⊙ Young diagram



$\lambda = (6, 4, 2, 2, 1)$

Length: $\ell(\lambda) = 5$

size: $|\lambda| = 6+4+2+2+1 = 15$
(or $\lambda \vdash 15$)

Symmetric monomial indexed by the partition $\lambda = (\lambda_1, \lambda_2, \dots)$

$$m_{(\lambda_1, \lambda_2, \dots)}(z_1, z_2, \dots) = \sum_{\sigma}{}' \, z_{\sigma(1)}^{\lambda_1} z_{\sigma(2)}^{\lambda_2} \cdots$$

where the sum is over permutations distinct monomials

E.g.: $m_{(3,1)}$ from the previous slide.

$\boxed{\text{Fact:}}$ $\{m_\lambda : \lambda \vdash d\}$ is a lin.-alg. basis for Λ^d.

Monomial positivity

Similarly, $\{m_\lambda : \lambda \in \mathrm{Par}\}$ is a basis for Λ,
where Par = space of all partitions

Def. $g \in \Lambda$ is <u>monomial-positive</u> (m-positive)

if $g = \sum_{\lambda \in \mathrm{Par}} c_\lambda m_\lambda$ with $c_\lambda \geq 0$ $\forall \lambda \in \mathrm{Par}$.

Fact. if g is m-positive, then it holds that

$$\boxed{|g(z_1, \ldots, z_n)| \leq g(|z_1|, \ldots, |z_n|)} \qquad (*)$$

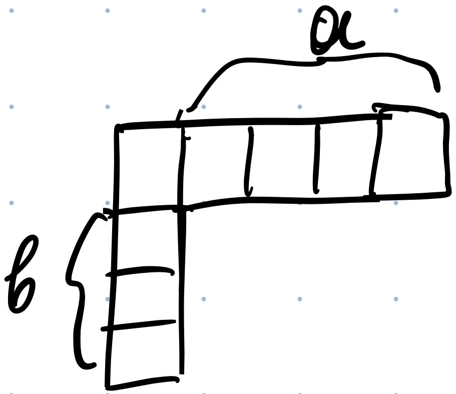
Corollary. Assume g is positive in some other <u>basis</u>, and all functions of this basis are m-positive. Then g is m-positive, and $(*)$ remains valid.

Schur functions

There is a more tractable basis: Schur functions s_λ.

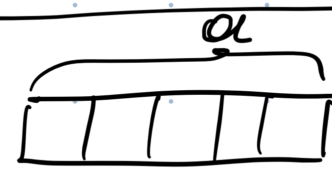
Fact s-positivity $\Rightarrow$ m-positivity. (See "Kostka numbers")

Hook-shaped partitions:



$(\alpha, 1^{b+1}) = (\alpha | b)$

Frobenius notation



$s_{(\alpha)} = e_\alpha$

$s_{(1^b)} = h_b$

Pieri rule: $h_a e_b = s_{(a|b-1)} + s_{(a-1|b)}$ $\iff$

$$s_{(a|b)} = \sum_{j=0}^{a} (-1)^j h_{a-j} e_{b+j+1} = \sum_{k=0}^{b} (-1)^k h_{a+k+1} e_{b-k}$$

Concluding the proof of Thm.

⊙ By Prop. 2:

$$\left|\frac{\psi_k^{(t)}}{k!}\right| = \left|\sum_{j=0}^{t-n} (-1)^{n-k+j} h_{t-n-j}(z_{1:n})\, e_{n-k+j}(z_{1:n})\right|$$

$$= \left| S_{(t-n|n-k-1)}(z_{1:n})\right| \leq S_{(t-n|n-k-1)}(|z_1|,\dots,|z_n|).$$

since S_λ is m-positive

⊙ As the result,

$$\left|\left\langle \psi_{0:n-1}^{(k)}, w_{1:n}\right\rangle\right| \leq \sum_{k\in[n]} w_k \left|S_{(t-n|n-k)}(|z_1|,\dots,|z_n|)\right|$$

$$\leq \sum_{k\in[n]} w_k\, S_{(t-n|n-k)}(r_{1:n}). \qquad \blacksquare$$

Thank You!

Omitted Proofs

Proof of interpolation lemmas

Proof of Lemma 2: Let $z \notin \{z_1, .., z_n\}$. Then $g(z) = f(z)\frac{g(z)}{f(z)}$

$$\hat{g}(z) = \sum_{k\in[n]} g(z_k) \prod_{j\neq k} \frac{z-z_j}{z_k - z_j} \qquad \text{(Lagrange form)}$$

$$= f(z) \sum_{k\in[n]} \frac{g(z_k)}{z-z_k} \underbrace{\prod_{j\neq k} \frac{1}{z_k - z_j}}_{= \left.\frac{\zeta - z_k}{f(\zeta)}\right|_{\zeta = z_k}} \qquad \text{(Barycentric form)}$$

$\Rightarrow$

$$\frac{g(z) - \hat{g}(z)}{f(z)} = \lim_{\zeta\to z} \overbrace{\frac{g(\zeta)}{f(\zeta)(\zeta - z)}}^{= F_z(\zeta)} (\zeta - z) + \sum_{k\in[n]} \lim_{\zeta \to z_k} \frac{g(\zeta)}{f(\zeta)(\zeta - z)} (\zeta - z_k)$$

$$= \operatorname{Res}(F_z, z) + \sum_{k\in[n]} \operatorname{Res}(F_z, z_k).$$

since $z, z_1, .., z_n$ are simple poles. Now invoke the residue thm ▨

Proof of interpolation lemmas

Proof of Lemma 1

- After transposition, the Vandermonde matrix

$$V_n[z_{1:n}] := \begin{pmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_n \\ \vdots & & \vdots \\ z_1^{n-1} & & z_n^{n-1} \end{pmatrix}$$ evaluates $\psi(z) = \sum_{k=0}^{n-1} \psi_k z^k$ on $z_{1:n}$:

$$V_n^T[z_{1:n}]\,\psi_{0:n-1} = \begin{bmatrix} \psi(z_1) \\ \vdots \\ \psi(z_n) \end{bmatrix} = \psi(z_{1:n}) \iff \boxed{\psi_{0:n-1} = V_n^{-T}[z_{1:n}]\,\psi(z_{1:n})}$$

- On the other hand, vectorization of the recurrence yields

$$x_{t:n+t-1}{}^T e_1 = (A_n^t(f)\, x_{0:n-1})^T e_1$$

companion mtx

$$= (V_n \Lambda_n^t V_n^{-1} x_{0:n-1})^T e_1$$

$$= x_{0:n-1}{}^T V_n^{-T} \Lambda_n^t V_n^T e_1$$

$$\Lambda_n^t V_n^T e_1 = \mathrm{Diag}(z_{1:n}^t)\,\mathbb{1}_n = z_{1:n}^t$$

$$= x_{0:n-1}{}^T \psi_{0:n-1}^{(t)}$$